Imperial College London

Lecture 15

Discrete-Time System Analysis using z-Transform

(Lathi 5.1)

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z-transform derived from Laplace transform

- Consider a discrete-time signal x(t) below sampled every T sec.

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + x_2\delta(t-2T) + x_3\delta(t-3T) + ...$$

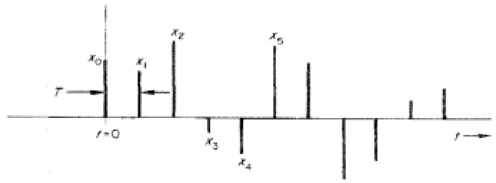
$$\delta(t) \Leftrightarrow 1 \quad (L6S5)$$
$$\delta(t-T) \Leftrightarrow e^{-sT} \quad (L6S13)$$

- The Laplace transform of x(t) is therefore (Time-shift prop. L6S13):

$$X(s) = x_0 + x_1 e^{-sT} + x_2 e^{-s2T} + x_3 e^{-s3T} + ...$$

- Now define $z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T}\cos\omega T + je^{\sigma T}\sin\omega T$

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + ...$$



L5.8 p560

z^{-1} the sample period delay operator

- From Laplace time-shift property, we know that $z = e^{sT}$ is time advance by T second (T is the sampling period).
- Therefore $z^{-1} = e^{-sT}$ corresponds to UNIT SAMPLE PERIOD DELAY.
- As a result, all sampled data (and discrete-time system) can be expressed in terms of the variable z.
- More formally, the **unilateral z-transform** of a causal sampled sequence: x[n] = x[0] + x[1] + x[2] + x[3] + is given by:

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + ...$$
$$= \sum_{n=0}^{\infty} x[n]z^{-n}$$

- The **bilateral z-transform** for a general sampled sequence is:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

L5.8 p560

Laplace, Fourier and z-Tranforms

	Definition	Purpose	Suitable for ..
Laplace transform	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	Converts **integral-differential** equations to **algebraic** equations	Continuous-time system & signal analysis; stable or unstable
Fourier transform	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Converts **finite time** signal to frequency domain	Continuous-time; stable system, convergent signals only; best for steady-state
Discrete Fourier transform	$X[n\omega_0] = \sum_{n=0}^{N_0-1} x[n]e^{jn\omega_0 T}$ N_0 samples, T = sample period $\omega_0 = 2\pi / T$	Converts finite **discrete-time** signal to **discrete frequency** domain	Discrete time, otherwise same as FT
z transform	$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	Converts **difference** equations into **algebraic** equations	Discrete-time system & signal analysis; stable or unstable

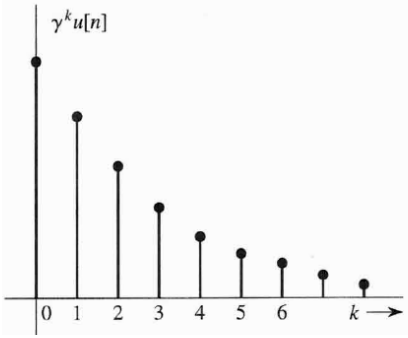
Example of z-transform (1)

- Find the z-transform for the signal $\gamma^n u[n]$, where γ is a constant.
- By definition $$X[z] = \sum_{n=0}^{\infty} \gamma^n u[n] z^{-n}$$
- Since u[n] = 1 for all n ≥ 0 (step function),
$$X[z] = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n = 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots + \cdots$$
- Apply the **geometric progression** formula:
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad \text{if} \quad |x| < 1$$
- Therefore:
$$X[z] = \frac{1}{1 - \frac{\gamma}{z}} \qquad \left|\frac{\gamma}{z}\right| < 1$$
$$= \frac{z}{z-\gamma} \qquad |z| > |\gamma|$$

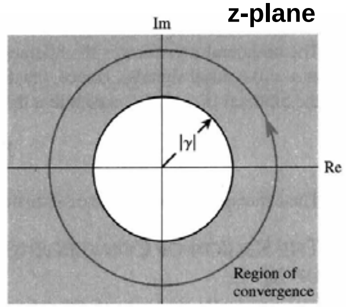
L5.1 p496

Example of z-transform (2)

- Observe that a simple equation in z-domain results in an infinite sequence of samples.
- Observe also that $X[z]$ exists only for $|z| > |\gamma|$.
- For $|z| < |\gamma|$, X[z] may go to infinity. We call the region of z-plane where X[z] exists as Region-of-Convergence (ROC), and is shown below.



z-plane



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z-transforms of δ[n] and u[n]

- Remember that by definition:
$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots$$
- Since $x[n] = \delta[n]$, $x[0] = 1$ and $x[2] = x[3] = x[4] = \cdots = 0$
$$\delta[n] \Longleftrightarrow 1 \qquad \text{for all } z$$
- Also, for $x[n] = u[n]$, $x[0] = x[1] = x[3] = \cdots = 1$.
- Therefore $$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$
$$= \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1 \qquad = \frac{z}{z-1} \qquad |z| > 1$$
$$u[n] \Longleftrightarrow \frac{z}{z-1} \qquad |z| > 1$$

L5.1 p499

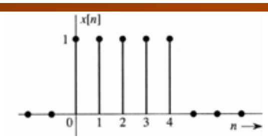
z-transforms of cosβn u[n]

- Since $\cos \beta n = (e^{j\beta n} + e^{-j\beta n})/2$
- From slide 5, we know $\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma} \qquad |z| > |\gamma|$
- Hence
$$e^{\pm j\beta n} u[n] \Longleftrightarrow \frac{z}{z - e^{\pm j\beta}} \qquad |z| > |e^{\pm j\beta}| = 1$$
- Therefore $$X[z] = \frac{1}{2}\left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}}\right]$$
$$= \frac{z(z - \cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1$$

L5.1 p500

z-transforms of 5 impulses

- Find the z-tranform of:
- By definition, $X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$



- Now remember the equation for sum of a power series:

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

- Let $r = z^{-1}$ and $n = 4$

$$X[z] = \frac{z^{-5} - 1}{z^{-1} - 1}$$

$$= \frac{z}{z-1}(1 - z^{-5})$$

L5.1 p500

z-transform Table (1)

No.	$x[n]$	$X[z]$
1	$\delta[n-n]$	z^{-k}
2	$u[n]$	$\frac{z}{z-1}$
3	$nu[n]$	$\frac{z}{(z-1)^2}$
4	$n^2u[n]$	$\frac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\frac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\frac{\gamma z}{(z-\gamma)^2}$

L5.1 p498

z-transform Table (2)

No.	$x[n]$	$X[z]$
10	$\frac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\frac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos\beta n\, u[n]$	$\frac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin\beta n\, u[n]$	$\frac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\frac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ $\gamma = \lvert\gamma\rvert e^{j\beta}$	$\frac{(0.5re^{j\theta})z}{z-\gamma} + \frac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\frac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r = \sqrt{\frac{A^2|\gamma|^2 + B^2 - 2AaB}{|\gamma|^2 - a^2}} \qquad \beta = \cos^{-1}\frac{-a}{|\gamma|} \qquad \theta = \tan^{-1}\frac{Aa - B}{A\sqrt{|\gamma|^2 - a^2}}$$

L5.1 p498

Inverse z-transform

- As with other transforms, inverse z-transform is used to derive x[n] from X[z], and is formally defined as:

$$x[n] = \frac{1}{2\pi j}\oint X[z]z^{n-1}\,dz$$

- Here the symbol $\oint$ indicates an integration in counterclockwise direction around a closed path in the complex z-plane (known as contour integral).
- Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the inverse z-tranform.
- One such technique is to use the z-transform pair table shown in the last two slides with partial fraction.

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Find inverse z-transform – real unique poles

- Find the inverse z-transform of: $X[z]=\dfrac{8z-19}{(z-2)(z-3)}$
- Step 1: Divide both sides by z: $\dfrac{X[z]}{z}=\dfrac{8z-19}{z(z-2)(z-3)}$
- Step 2: Perform partial fraction: $\dfrac{X[z]}{z}=\dfrac{(-19/6)}{z}+\dfrac{(3/2)}{z-2}+\dfrac{(5/3)}{z-3}$
- Step 3: Multiply both sides by z: $X[z]=-\dfrac{19}{6}+\dfrac{3}{2}\left(\dfrac{z}{z-2}\right)+\dfrac{5}{3}\left(\dfrac{z}{z-3}\right)$
- Step 4: Obtain inverse z-transform of each term from table (#1 & #6):

$$x[n]=-\tfrac{19}{6}\delta[n]+\left[\tfrac{3}{2}(2)^n+\tfrac{5}{3}(3)^n\right]u[n]$$

L5.1-1 p501

Find inverse z-transform – repeat real poles (1)

- Find the inverse z-transform of: $X[z]=\dfrac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$
- Divide both sides by z and expand:

$$\frac{X[z]}{z}=\frac{2z^2-11z+12}{(z-1)(z-2)^3}=\frac{k}{z-1}+\frac{a_0}{(z-2)^3}+\frac{a_1}{(z-2)^2}+\frac{a_2}{(z-2)}$$

- Use covering method to find k and a_0:

$$k=\left.\frac{2z^2-11z+12}{(z-1)(z-2)^3}\right|_{z=1}=-3 \qquad a_0=\left.\frac{2z^2-11z+12}{(z-1)(z-2)^3}\right|_{z=2}=-2$$

- We get:

$$\frac{X[z]}{z}=\frac{2z^2-11z+12}{(z-1)(z-2)^3}=\frac{-3}{z-1}-\frac{2}{(z-2)^3}+\frac{a_1}{(z-2)^2}+\frac{a_2}{(z-2)}$$

- To find a_2, multiply both sides by z and let $z\to\infty$:

$$0=-3-0+0+a_2 \implies a_2=3$$

L5.1-1 p502

Find inverse z-transform – repeat real poles (2)

- To find a_1, let z = 0:

$$\frac{12}{8}=3+\frac{1}{4}+\frac{a_1}{4}-\frac{3}{2} \implies a_1=-1$$

- Therefore, we find:

$$\frac{X[z]}{z}=\frac{-3}{z-1}-\frac{2}{(z-2)^3}-\frac{1}{(z-2)^2}+\frac{3}{z-2}$$

$$X[z]=-3\frac{z}{z-1}-2\frac{z}{(z-2)^3}-\frac{z}{(z-2)^2}+3\frac{z}{z-2}$$

- Use pairs #6 & #10

6	$\gamma^n u[n]$	$\Longleftrightarrow$	$\dfrac{z}{z-\gamma}$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\Longleftrightarrow$	$\dfrac{z}{(z-\gamma)^{m+1}}$

$$x[n]=\left[-3-2\frac{n(n-1)}{8}(2)^n-\frac{n}{2}(2)^n+3(2)^n\right]u[n]$$

$$=-\left[3+\frac{1}{4}(n^2+n-12)2^n\right]u[n]$$

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Find inverse z-transform – complex poles (1)

- Find inverse z-tranform of: $X[z]=\dfrac{2z(3z+17)}{(z-1)(z^2-6z+25)}=\dfrac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$
- Whenever we encounter complex pole, we need to use a special partial fraction method (called quadratic factors):

$$\frac{X[z]}{z}=\frac{2(3z+17)}{(z-1)(z^2-6z+25)}=\frac{2}{z-1}+\frac{Az+B}{z^2-6z+25}$$

- Now multiply both sides by z, and let $z\to\infty$:

$$0=2+A \Longrightarrow A=-2$$

- We get:

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)}=\frac{2}{z-1}+\frac{-2z+B}{z^2-6z+25}$$

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Find inverse z-transform – complex poles (2)

- To find B, we let z=0:

$$\frac{-34}{25} = -2 + \frac{B}{25} \Longrightarrow B = 16$$

- Now, we have X[z] in a convenient form:

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25} \quad \Longrightarrow \quad X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

- Use table pair #12c, we identify A = -2, B = 16, $|\gamma| = 5$, and a = -3.

$$r = \sqrt{\frac{100+256-192}{25-9}} = 3.2, \quad \beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927 \text{ rad}$$

$$\theta = \tan^{-1}\left(\tfrac{-10}{-8}\right) = -2.246 \text{ rad}$$

12c	$r\|\gamma\|^n \cos(\beta n + \theta)u[n]$	$\Longleftrightarrow$	$\frac{z(Az+B)}{z^2+2az+\|\gamma\|^2}$

- Therefore:

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$$

L5.1-1 p504

Find inverse z-transform – long division

- Consider this example:

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = \frac{7z^3-2z^2}{z^3-1.7z^2+0.8z-0.1}$$

- Perform long division:

$$\begin{array}{r} 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots \\ z^3 - 1.7z^2 + 0.8z - 0.1 \,\big)\, 7z^3 - 2z^2 \qquad\qquad\qquad\qquad \\ 7z^3 - 11.9z^2 + 5.60z - 0.7 \qquad\qquad \\ 9.9z^2 - 5.60z + 0.7 \qquad\qquad \\ 9.9z^2 - 16.83z + 7.92 - 0.99z^{-1} \\ 11.23z - 7.22 + 0.99z^{-1} \\ 11.23z - 19.09 + 8.98z^{-1} \\ 11.87 - 7.99z^{-1} \end{array}$$

- Thus:

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

- Therefore

$$x[0] = 7, \; x[1] = 9.9, \; x[2] = 11.23, \; x[3] = 11.87, \ldots$$

L5.1-1 p505